

RELATIONS BETWEEN SOME INVARIANTS OF ALGEBRAIC VARIETIES IN POSITIVE CHARACTERISTIC

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ABSTRACT. We discuss relations between certain invariants of varieties in positive characteristic, like the a -number and the height of the Artin-Mazur formal group. We calculate the a -number for Fermat surfaces.

1. INTRODUCTION

Algebraic varieties in positive characteristic possess special invariants that have no analogue in characteristic 0. In this paper we consider three such invariants related to the cohomology groups H^n with n equal to the dimension of the variety. The first, the a -number that we introduced in [6], registers where the image of Frobenius acting on $H^n(X, \mathcal{O}_X)$ lands in the Hodge filtration of $H_{\text{dR}}^n(X)$. For abelian varieties it coincides with the a -number defined by Oort. The second one is the height $h(X)$ of the Artin-Mazur formal group, which is an infinitesimal invariant related to the étale cohomology group $H_{\text{et}}^n(X, \mathbf{G}_m)$. The third, baptised the b -number $b(X)$, is related to $H^n(X, B_i)$ with B_i the sheaves of exact 1-forms defined by Illusie.

These invariants are related in subtle ways. In this note we prove the relation

$$h(X) = b(X) + p_g(X)$$

and in case $b(X) < \infty$ the basic estimate

$$b(X) \leq \dim H^{n-1}(X, \Omega_X^1).$$

Furthermore we prove that if $b(X) < \infty$ then $a(X) = 0$ or $a(X) = 1$ and we show that $a = 0$ if and only if $b = 0$. As an example we calculate the a -number for Fermat surfaces.

Throughout this paper k denotes an algebraically closed field of characteristic $p > 0$ and all varieties considered are complete and non-singular.

2. THE a -NUMBER

Let X be a complete non-singular variety of dimension n over k . The de Rham cohomology of X is the hypercohomology of the complex (Ω_X^\bullet, d) . We are interested in the n -th de Rham cohomology group and its Hodge filtration

$$H_{\text{dR}}^n(X) = F^0 \supset F^1 \supset \cdots \supset F^n \supset (0).$$

We shall assume in this section that the Hodge-to-de Rham spectral sequence

$$E_1^{i,j} = H^j(X, \Omega_X^i) \Longrightarrow H_{\text{dR}}^{i+j}(X)$$

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degenerates at the E_1 -level. (This condition is for example fulfilled if the characteristic p satisfies $p > n$ and if X can be lifted to the Witt ring $W_2(k)$ of length 2, cf. Deligne-Illusie [3].) In this case the graded pieces are

$$H^n(X, \mathcal{O}_X) \cong F^0/F^1, H^{n-1}(X, \Omega_X^1) \cong F^1/F^2, \dots, H^0(X, \Omega_X^n) \cong F^n.$$

We have a Frobenius morphism F acting on $H_{\text{dR}}^n(X)$; it acts by 0 on F^1 and it induces a homomorphism

$$F : H^n(X, \mathcal{O}_X) \cong F^0/F^1 \longrightarrow H_{\text{dR}}^n(X).$$

By Katz [10] this induced homomorphism is injective.

In [6] we defined an invariant called the a -number as follows.

Definition 2.1. The a -number of the variety X is defined by

$$a(X) = \max\{i : (\text{Im } F) \cap F^i \neq (0)\}.$$

Note that $0 \leq a(X) \leq n = \dim(X)$.

For an abelian variety X Oort defined in [14] the a -number $a(X)$ by

$$a(X) = \dim_k \text{Hom}(\alpha_p, X),$$

with α_p the local-local group scheme (that is, the kernel of Frobenius acting on the additive group \mathbf{G}_a). Since $\text{End}(\alpha_p) \cong k$ we can view $\text{Hom}(\alpha_p, X)$ as a right vector space over k . The union of all the images of the group scheme homomorphisms $\alpha_p \rightarrow X$ is the maximal subgroup scheme $A(X)$ of $X[p]$ annihilated by the operators F (Frobenius) and V (Verschiebung) on the kernel $X[p]$ of multiplication by p . Then this a -number is $\log_p \text{ord } A(X)$. The Dieudonné module of $X[p]$ can be identified with $H_{\text{dR}}^1(X)$ and the Dieudonné module of $A(X)$ can be identified with the kernel of V acting on $H^0(X, \Omega_X^1)$, the kernel of F . Since $\ker V = \text{Im } F$ Oort's a -number equals the dimension of the image of F (acting on $H_{\text{dR}}^1(X)$) in $H^0(X, \Omega_X^1)$:

$$\dim_k \text{Hom}(\alpha_p, X) = \dim H^0(X, \Omega_X^1) \cap (\text{Im } F).$$

We showed in [6] that for an abelian variety our definition of the a -number that involves $H_{\text{dR}}^n(X)$ coincides with Oort's definition that involves $H_{\text{dR}}^1(X)$. For the reader's convenience we recall the proof.

Proposition 2.2. *For an abelian variety X/k the two definitions of the a -number coincide.*

Proof. Recall that $H_{\text{dR}}^n(X) = \wedge^n H_{\text{dR}}^1(X)$ and if we write $H_{\text{dR}}^1(X) = V_1 \oplus V_2$ with $V_1 = H^0(X, \Omega_X^1)$ and V_2 a complementary subspace, then the Hodge filtration on $H_{\text{dR}}^n(X)$ is $F^r = \sum_{j=r}^n \wedge^j V_1 \otimes \wedge^{n-j} V_2$. We have $F(H^n(X, \mathcal{O}_X)) = F(\wedge^n H^1(X, \mathcal{O}_X)) = \wedge^n F(H^1(X, \mathcal{O}_X))$. If we write $F(H^1(X, \mathcal{O}_X)) = A \oplus B$ with A the intersection with $H^0(X, \Omega_X^1)$ and B a complementary space, then $\wedge^n(A \oplus B) = \wedge^a A \otimes \wedge^{n-a} B$ with $a = \dim(A)$. It follows that $F(H^n(X, \mathcal{O}_X))$ lies in F^a , but not in F^{a+1} . \square

The a -number is just one of the invariants that can be associated to the relative position of the Hodge filtration and the conjugate filtration on $H_{\text{dR}}^n(X)$, cf. [6], [12].

3. THE h -NUMBER

For a non-singular proper variety X/k of dimension n one can consider the formal completion of the Picard group. For any local artinian scheme S with residue field k its S -valued points are given by the exact sequence

$$0 \rightarrow \widehat{\text{Pic}}(X)(S) \rightarrow H_{\text{et}}^1(X \times S, \mathbf{G}_m) \rightarrow H_{\text{et}}^1(X, \mathbf{G}_m),$$

where \mathbf{G}_m denotes the multiplicative group. This invariant provides interesting information, for example for elliptic curves in positive characteristic. This idea was generalized by Artin and Mazur to the higher cohomology groups in [2]. Let \mathcal{C} the category of Artinian local algebras (R, m) over k with maximal ideal m such that $R/m \cong k$, and denote by \mathcal{A} the category of abelian groups. We consider the covariant functor $\Phi_X = \Phi_X^{(n)} : \mathcal{C} \rightarrow \mathcal{A}$ defined by

$$\Phi_X(R) = \ker(H_{\text{et}}^n(X \times_k \text{Spec } R, \mathbf{G}_m) \rightarrow H_{\text{et}}^n(X, \mathbf{G}_m))$$

for R an object in \mathcal{C} . Here $H_{\text{et}}^n(X, \mathbf{G}_m)$ denotes the n -th étale cohomology group with values in the multiplicative group \mathbf{G}_m , and $X \rightarrow X \times_k \text{Spec } R$ is the natural immersion. When the functor Φ_X is pro-representable by a formal Lie group, we call the formal Lie group an Artin-Mazur formal group. We also denote by Φ_X the formal Lie group. The tangent space of Φ_X is given by $H^n(X, \mathcal{O}_X)$ (cf. Artin-Mazur [2]). If Φ_X is pro-representable by a formal Lie group we denote by $h(X)$ the height of the corresponding formal Lie group Φ_X and call $h(X)$ the h -number of X .

This h -number is a special invariant in positive characteristic and $h(X)$ is either a positive integer or ∞ . For example, in the case of an elliptic curve the h -number assumes the values 1 or 2 depending on whether the elliptic curve is ordinary or supersingular; for a K3 surface X we know that either $1 \leq h(X) \leq 10$ or $h(X) = \infty$, the latter if Φ_X is the additive group \mathbf{G}_a , cf. Artin [1, 2].

4. THE b -NUMBER

Let X be a complete non-singular variety over k . Following Illusie ([9]) we define $B_1 = d\mathcal{O}_X$ and $Z^1 = \ker(\Omega_X^1 \xrightarrow{d} \Omega_X^2)$. Using the Cartier operator

$$C : Z^1 \rightarrow \Omega_X^1$$

we can define inductively sheaves of \mathcal{O}_X -modules for $j \geq 2$ by

$$\begin{aligned} Z^j &= C^{-1}(Z^{j-1}), \\ B_j &= C^{-1}(B_{j-1}). \end{aligned}$$

Since $B_1 \subseteq Z^1$ we get a filtration

$$0 = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_i \subset \cdots \subset Z^i \subset \cdots \subset Z^1 \subset Z^0 = \Omega_X^1.$$

Recall that the sheaf B_i admits a description in terms of Witt vector cohomology by the Serre differential $D_i : W_i(\mathcal{O}_X) \rightarrow B_i$ given by

$$(a_0, \dots, a_{i-1}) \mapsto a_0^{p^{i-1}-1} da_0 + a_1^{p^{i-2}-1} da_1 + \cdots + a_{i-2}^{p-1} da_{i-2} + da_{i-1}.$$

This map D_i induces an isomorphism

$$W_i(\mathcal{O}_X)/FW_i(\mathcal{O}_X) \cong B_i,$$

where F is the Frobenius operator, cf. Serre [16]. The cohomology of these sheaves B_i and Z^i leads to interesting invariants. One example is:

Definition 4.1. The b -number of X is $b(X) = \max_{i \geq 1} \dim H^n(X, B_i)$.

By the exact sequence

$$0 \rightarrow B_1 \longrightarrow B_{i+1} \xrightarrow{C} B_i \rightarrow 0$$

we have a surjective homomorphism

$$C : H^n(X, B_{i+1}) \longrightarrow H^n(X, B_i),$$

which gives a projective system $\{C, H^n(X, B_i)\}$ and we may reformulate the definition of $b(X)$ as

$$b(X) = \dim \varprojlim H^n(X, B_i).$$

The b -number can be related to the action of Frobenius on Witt vector cohomology as follows.

Proposition 4.2. We have $b(X) = \dim_k H^n(X, W(\mathcal{O}_X))/FH^n(X, W(\mathcal{O}_X))$.

Note that $H^n(X, W(\mathcal{O}_X))/FH^n(X, W(\mathcal{O}_X))$ is a vector space over $W(k)/pW(k) \cong k$.

Proof. We have the commutative diagram

$$\begin{array}{ccc} W_{i+1} & \xrightarrow{D_{i+1}} & B_{i+1} \\ \downarrow R & & \downarrow C \\ W_i & \xrightarrow{D_i} & B_i \end{array}$$

and a map of exact sequences

$$\begin{array}{ccccccc} H^n(X, W_{i+1}(\mathcal{O}_X)) & \xrightarrow{F} & H^n(X, W_{i+1}(\mathcal{O}_X)) & \longrightarrow & H^n(X, B_{i+1}) & \longrightarrow & 0 \\ \downarrow R & & \downarrow R & & \downarrow C & & \\ H^n(X, W_i(\mathcal{O}_X)) & \xrightarrow{F} & H^n(X, W_i(\mathcal{O}_X)) & \longrightarrow & H^n(X, B_i) & \longrightarrow & 0 \end{array}$$

The projective system $\{R : H^n(X, W_{i+1}(\mathcal{O}_X)) \rightarrow H^n(X, W_i(\mathcal{O}_X))\}$ satisfies the Mittag-Leffler condition, and by taking the projective limit we obtain an exact sequence

$$H^n(X, W(\mathcal{O}_X)) \xrightarrow{F} H^n(X, W(\mathcal{O}_X)) \longrightarrow \varprojlim H^n(X, B_i) \rightarrow 0,$$

and this gives us the desired conclusion. \square

Another characterization of the b -number uses the exact sequence

$$0 \rightarrow W_i(\mathcal{O}_X) \xrightarrow{F} W_i(\mathcal{O}_X) \xrightarrow{D_i} B_i \rightarrow 0$$

and the induced long exact cohomology sequence.

Proposition 4.3. We have $b(X) = \max_{i \geq 1} \dim H^{n-1}(X, B_i)/\text{Im } D_i$.

Proof. From the long exact cohomology sequence we get the exact sequence

$$\begin{aligned} 0 \rightarrow H^{n-1}(X, B_i)/\text{Im } D_i &\rightarrow H^n(X, W_i(\mathcal{O}_X)) \xrightarrow{F} H^n(X, W_i(\mathcal{O}_X)) \\ &\xrightarrow{D_i} H^n(X, B_i) \rightarrow 0. \end{aligned}$$

Looking at the lengths of these $W_i(k)$ -modules, we observe

$$\dim H^{n-1}(X, B_i)/\text{Im } D_i = \dim H^n(X, B_i).$$

□

5. AN INEQUALITY FOR THE b -NUMBER

We now will prove a basic inequality for the b -number.

Theorem 5.1. *If $b(X) < \infty$ then $b(X) \leq \dim H^{n-1}(X, \Omega_X^1)$.*

The natural inclusion $B_\ell \rightarrow \Omega_X^1$ induces a homomorphism

$$\varphi_\ell : H^{n-1}(X, B_\ell) \longrightarrow H^{n-1}(X, \Omega_X^1).$$

The proof of this theorem relies on the following lemma relating the kernel of φ_ℓ and the image of D_ℓ .

Lemma 5.2. *If $b(X) < \infty$ then $\ker \varphi_\ell \subset \text{Im } D_\ell$ for any positive integer ℓ .*

Proof. Assuming that the result does not hold we consider the smallest positive ℓ such that $\ker \varphi_\ell \not\subset \text{Im } D_\ell$. Then there exists a non-zero element $\alpha \in H^{n-1}(X, B_\ell)$ such that $\varphi_\ell(\alpha) = 0$ and $\alpha \notin \text{Im } D_\ell$. Let m be the non-negative integer such that $C^m(\alpha) \notin \text{Im } D_{\ell-m}$ and $C^{m+1}(\alpha) \in \text{Im } D_{\ell-m-1}$. Here we define $\text{Im } D_k = 0$ for $k \leq 0$. For any positive integer s the commutativity of the diagram

$$\begin{array}{ccc} H^{n-1}(X, W_{i+1}) & \xrightarrow{D_{i+1}} & H^{n-1}(X, B_{i+1}) \\ \downarrow R & & \downarrow C \\ H^{n-1}(X, W_i) & \xrightarrow{D_i} & H^{n-1}(X, B_i) \end{array}$$

implies $C^{m+s}(\alpha) \in \text{Im } D_{\ell-m-s}$

We take an affine open covering $\{U_i\}$ of X . Then α is given by a Čech cocycle $\{\alpha_I\}$ with $I = i_0 i_1 \cdots i_{n-1}$ and

$$\alpha_I = \sum_{j=0}^{\ell-1} (f_I^{(j)})^{p^{\ell-1-j}-1} df_I^{(j)}$$

for $f_I^{(j)} = f_{i_0 i_1 \cdots i_{n-1}}^{(j)} \in \Gamma(U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_{n-1}}, \mathcal{O}_X)$. Since by assumption $\varphi_\ell(\alpha) = 0$, there exist elements $\omega_{i_0 i_1 \cdots i_{n-2}} \in \Gamma(U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_{n-2}}, \Omega_X^1)$ such that

$$\alpha_I = \omega_{i_1 i_2 \cdots i_{n-1}} - \omega_{i_0 i_2 \cdots i_{n-1}} + \cdots + (-1)^{n-1} \omega_{i_0 i_1 \cdots i_{n-2}} \quad (1)$$

For an affine open set U , we have $H^1(U, B_1) = 0$. Thus the exact sequence

$$0 \rightarrow B_1 \rightarrow Z^1 \xrightarrow{C} \Omega_X^1 \rightarrow 0$$

implies that the Cartier operator $C : \Gamma(U, Z^1) \rightarrow \Gamma(U, \Omega_X^1)$ is surjective. So we can find an element $\tilde{\omega}_{i_0 i_1 \cdots i_{n-2}} \in \Gamma(U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_{n-2}}, \Omega_X^1)$ that maps to $\omega_{i_0 i_1 \cdots i_{n-2}}$ under C and then

$$\tilde{\omega}_{i_1 i_2 \cdots i_{n-1}} - \tilde{\omega}_{i_0 i_2 \cdots i_{n-1}} + \cdots + (-1)^{n-1} \tilde{\omega}_{i_0 i_1 \cdots i_{n-2}}$$

maps under C to the right hand side of (1). Since $\omega \in \Gamma(U, Z^1)$ has $C(\omega) = 0$ if and only if $\omega = df$ for a suitable regular function $f \in \Gamma(U, \mathcal{O}_X)$, we can choose a regular function $f_I^{(\ell)} = f_{i_0 i_1 \dots i_{n-1}}^{(\ell)} \in \Gamma(U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_{n-1}}, \mathcal{O}_X)$ such that

$$\sum_{j=0}^{\ell} (f_I^{(j)})^{p^{\ell-j}-1} df_I^{(j)} = \tilde{\omega}_{i_1 i_2 \dots i_{n-1}} - \tilde{\omega}_{i_0 i_2 \dots i_{n-1}} + \dots + (-1)^{n-1} \tilde{\omega}_{i_0 i_1 \dots i_{n-2}}$$

In this way the cochain $C^{-1}(\alpha) = \sum_{j=0}^{\ell} (f_I^{(j)})^{p^{\ell-j}-1} df_I^{(j)}$ becomes a co-cycle in the Čech co-chains of the sheaf $B_{\ell+1}$ and gives an element of $H^{n-1}(X, B_{\ell+1})$. Repeating this procedure t times we obtain t elements

$$\alpha, C^{-1}(\alpha), C^{-2}(\alpha), \dots, C^{-t}(\alpha)$$

in $H^{n-1}(X, B_{\ell+t})$.

Consider the vector space $H^{n-1}(X, B_{\ell+t})/\text{Im } D_{\ell+t}$ over k . Note that the Cartier operator $C : B_{\ell+i+1} \rightarrow B_{\ell+i}$ induces a p^{-1} -linear mapping

$$C : H^{n-1}(X, B_{\ell+i+1})/\text{Im } D_{\ell+i+1} \longrightarrow H^{n-1}(X, B_{\ell+i})/\text{Im } D_{\ell+i}.$$

Suppose the elements $\alpha, C^{-1}(\alpha), C^{-2}(\alpha), \dots, C^{-t}(\alpha)$ are linearly dependent over k in $H^{n-1}(X, B_{\ell+t})/\text{Im } D_{\ell+t}$. So there exist elements $a_i \in k$ with $i = 0, 1, \dots, t$ such that

$$a_0 \alpha + a_1 C^{-1}(\alpha) + a_2 C^{-2}(\alpha) + \dots + a_t C^{-t}(\alpha) = 0$$

in $H^{n-1}(X, B_{\ell+t})/\text{Im } D_{\ell+t}$. By letting C^{t+m} operate on both sides we have

$$a_t^{p^{-t-m}} C^m(\alpha) = 0$$

in $H^{n-1}(X, B_{\ell-m})/\text{Im } D_{\ell-m}$. By our assumption, $C^m(\alpha)$ is not contained in $\text{Im } D_{\ell-m}$. It follows that $a_t = 0$. Repeating this procedure we see $a_0 = a_1 = \dots = a_t = 0$. This means that our elements are linearly independent over k and we see that $\dim H^{n-1}(X, B_{\ell+t})/\text{Im } D_{\ell+t} \geq t+1$ for any positive integer t , which contradicts the finiteness of $b(X)$. We conclude that $\ker \varphi_\ell \subset \text{Im } D_\ell$ for any positive integer ℓ . \square

We now prove Theorem 5.1. We have by Proposition 4.3 and its proof and by Lemma 5.2 that

$$\dim H^n(X, B_i) = \dim H^{n-1}(X, B_i)/\text{Im } D_i \leq \dim H^{n-1}(X, B_i)/\ker \varphi_i$$

and since $\dim H^{n-1}(X, B_i)/\ker \varphi_i \leq \dim H^{n-1}(X, \Omega_X^1)$, we derive the inequality $b(X) \leq \dim H^{n-1}(X, \Omega_X^1)$. This concludes the proof of Theorem 5.1.

We conclude this section with a remark about the spaces $H^{n-1}(X, B_i)$. Using the natural inclusion $\psi_i : B_i \rightarrow B_{i+1}$ we have the induced linear mapping

$$\psi_i : H^{n-1}(X, B_i) \longrightarrow H^{n-1}(X, B_{i+1}).$$

Corollary 5.3. *Assume $b(X) < \infty$. If for all i the map $D_i : H^{n-1}(X, W_i(\mathcal{O}_X)) \rightarrow H^{n-1}(X, B_i)$ is zero, then $\psi_i : H^{n-1}(X, B_i) \rightarrow H^{n-1}(X, B_{i+1})$ is injective for any $i \geq 1$. In particular, if $H^{n-1}(X, \mathcal{O}_X) = 0$ then ψ_i is injective for any $i \geq 1$.*

Proof. The first part of this corollary follows from the fact that the composition of the homomorphisms

$$H^{n-1}(X, B_i) \xrightarrow{\psi_i} H^{n-1}(X, B_{i+1}) \xrightarrow{\varphi_{i+1}} H^{n-1}(X, \Omega_X^1),$$

where $\varphi_i = \varphi_{i+1} \circ \psi_i$, is injective. We have an exact sequence

$$0 \rightarrow W_{i-1}(\mathcal{O}_X) \xrightarrow{V} W_i(\mathcal{O}_X) \xrightarrow{R^{i-1}} \mathcal{O}_X \rightarrow 0.$$

So with $H^{n-1}(X, \mathcal{O}_X) = 0$ we find inductively $H^{n-1}(X, W_i(\mathcal{O}_X)) = 0$. Thus the second statement follows from the first one. \square

6. THE RELATION BETWEEN THE b -NUMBER AND THE h -NUMBER

Assuming in this section that the Artin-Mazur formal group $\Phi_X = \Phi_X^{(n)}$ is pro-representable by a formal Lie group we establish a relation between the h -number (=height) and the b -number.

Theorem 6.1. *Let X be a non-singular complete algebraic variety with the Artin-Mazur formal group Φ_X pro-representable by a formal Lie group and let $h(X)$ be the height of Φ_X . Then we have the equality*

$$h(X) = b(X) + p_g(X),$$

with $p_g(X) = \dim H^n(X, \mathcal{O}_X)$ the geometric genus.

Proof. This follows on the one hand from our interpretation of the b -number in terms of Witt vector cohomology (Lemma 4.2) and on the other hand by Dieudonné theory that expresses the height in terms of Witt vector cohomology as follows. We use the covariant Dieudonné module theory (Cartier Dieudonné module theory). For our Artin-Mazur formal group, the Dieudonné module is given by $H^n(X, W(\mathcal{O}_X))$. So the general theory of Dieudonné modules implies

$$h(X) = \dim H^n(X, W(\mathcal{O}_X))/pH^n(X, W(\mathcal{O}_X)).$$

Since $p = VF$ we have an exact sequence

$$\begin{aligned} 0 \rightarrow VH^n(X, W(\mathcal{O}_X))/pH^n(X, W(\mathcal{O}_X)) &\longrightarrow H^n(X, W(\mathcal{O}_X))/pH^n(X, W(\mathcal{O}_X)) \\ &\longrightarrow H^n(X, W(\mathcal{O}_X))/VH^n(X, W(\mathcal{O}_X)) \rightarrow 0 \end{aligned}$$

and we now have to calculate the dimensions of the second and fourth term in this sequence. By general Dieudonné module theory the Verschiebung V acting on $H^n(X, W(\mathcal{O}_X))$ is injective so that $p = VF$ implies

$$H^n(X, W(\mathcal{O}_X))/FH^n(X, W(\mathcal{O}_X)) \cong VH^n(X, W(\mathcal{O}_X))/pH^n(X, W(\mathcal{O}_X))$$

and we know by Proposition 4.2 that its dimension is $b(X)$. As to the dimension of the fourth term we observe that the exact sequence

$$0 \rightarrow VH^n(X, W(\mathcal{O}_X)) \longrightarrow H^n(X, W(\mathcal{O}_X)) \longrightarrow H^n(X, \mathcal{O}_X) \rightarrow 0$$

shows that the dimension of the fourth term is $\dim H^n(X, \mathcal{O}_X) = p_g(X)$. \square

By Theorem 5.1 we get the following upper bound on the h -number.

Corollary 6.2. *If $b(X) < \infty$ we have*

$$h(X) \leq \dim H^{n-1}(X, \Omega_X^1) + p_g(X).$$

The following corollary was already obtained in [7]. In view of its interest we state it in our new framework.

Corollary 6.3. *Let X be a Calabi-Yau variety of dimension $n \geq 1$. Then if $h(X) < \infty$ we have*

$$h(X) \leq \dim H^{n-1}(X, \Omega_X^1) + 1$$

In particular, if X is rigid, then $h(X) = 1$ or ∞ .

Proof. Note that Theorem 5.1 implies that $b(X) < \infty$. Since $\dim H^n(X, \mathcal{O}_X) = 1$, the first inequality follows from Corollary 6.2. If X is rigid then $H^1(X, \Theta_X) = 0$ by definition and by $H^1(X, \Theta_X) \cong H^{n-1}(X, \Omega_X^1)$, the conclusion follows from the inequality. \square

7. RELATIONS BETWEEN THE a -NUMBER AND THE b -NUMBER

In this section we shall assume that the Hodge-to de Rham spectral sequence degenerates at the E_1 -term.

Theorem 7.1. *If $b(X) < \infty$ then $a(X) = 0$ or $a(X) = 1$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} H^{n-1}(X, \mathcal{O}_X) & & \\ \downarrow D_1 & \searrow d & \\ H^{n-1}(X, B_1) & \xrightarrow{\varphi_1} & H^{n-1}(X, \Omega_X^1). \end{array}$$

Since by our assumption the Hodge-to-de Rham spectral sequence degenerates at the E_1 -term the map d is zero. Since by Lemma 5.2 the kernel $\ker \varphi_1$ is contained in the image $\text{Im } D_1$ we see that $\ker \varphi_1 = \text{Im } D_1$ and we thus have an injective homomorphism

$$H^{n-1}(X, B_1)/\text{Im } D_1 \hookrightarrow H^{n-1}(X, \Omega_X^1).$$

If $a(X) \geq 1$ then there exists a non-zero element $\alpha \in H^n(X, \mathcal{O}_X)$ such that $F(\alpha) \in F^1$. This means we have an element $F(\alpha) \in F^1/F^2 \cong H^{n-1}(X, \Omega_X^1)$. The exact sequence

$$0 \rightarrow \mathcal{O}_X \xrightarrow{F} \mathcal{O}_X \rightarrow B_1 \rightarrow 0,$$

gives rise to the long exact sequence

$$\rightarrow H^{n-1}(X, \mathcal{O}_X) \xrightarrow{D_1} H^{n-1}(X, B_1) \xrightarrow{\delta} H^n(X, \mathcal{O}_X) \xrightarrow{F} H^n(X, \mathcal{O}_X).$$

Now $F(\alpha) = 0$ in $F^0/F^1 \cong H^n(X, \mathcal{O}_X)$ and this tells us that there exists an element $\beta \in H^{n-1}(X, B_1)$ such that $\delta(\beta) = \alpha$. Since $\alpha \neq 0$, we see $\beta \notin \text{Im } D_1$. Using Čech cohomology, it is easy to see that $F(\alpha) = \varphi_1(\beta)$ in $H^{n-1}(X, \Omega_X^1)$. Since $F(\alpha) = \varphi_1(\beta) \neq 0$, we see $F(\alpha) \neq 0$ in $H^{n-1}(X, \Omega_X^1)$. Therefore, we have $F(\alpha) \notin F^2$ and it follows that $a(X) = 1$. \square

In particular, if the h -number is defined then $b(X) < \infty$ implies $h(X) < \infty$ and then for K3 surfaces and abelian varieties our result implies results like those in [13], [4], [5] and [6], Proposition 9.4.

Proposition 7.2. *For the variety X we have: $a(X) = 0$ if and only if $b(X) = 0$.*

Proof. If $a(X) = 0$ then in the Hodge filtration we have $\text{Im } F \cap F^1 = (0)$ resulting in an isomorphism

$$F : H^n(X, \mathcal{O}_X) \cong F^0/F^1 \cong H^n(X, \mathcal{O}_X).$$

Therefore the exact sequence

$$0 \rightarrow \mathcal{O}_X \xrightarrow{F} \mathcal{O}_X \xrightarrow{d} B_1 \rightarrow 0 \quad (1)$$

implies that $H^n(X, B_1) = (0)$. The exact sequence

$$0 \rightarrow B_1 \rightarrow B_{i+1} \xrightarrow{C} B_i \rightarrow 0$$

gives inductively $H^n(X, B_i) = (0)$ for all $i > 0$. This implies that $b(X) = 0$.

Conversely, if $b(X) = 0$ then in particular $H^n(X, B_1) = (0)$. By the exact sequence (1) we see that $F : H^n(X, \mathcal{O}_X) \rightarrow H^n(X, \mathcal{O}_X)$ is surjective and thus an isomorphism. Since $H^n(X, \mathcal{O}_X) \cong F^0/F^1$, we conclude that $(\text{Im } F) \cap F^1 = (0)$ and so $a(X) = 0$. \square

8. FERMAT SURFACES

As an example we now calculate the a -number of Fermat surfaces and deduce consequences for the h -number. Recall that a non-singular complete curve C is said to be *ordinary* if its Jacobian variety is an ordinary abelian variety. Note that a curve C is ordinary if and only if Frobenius induces a bijective map from $H^1(C, \mathcal{O}_C)$ to its image. Hence $a(C) = 0$ if C is ordinary and $a(C) = 1$ otherwise.

Proposition 8.1. *Let C_1 and C_2 be non-singular complete algebraic curves defined over k . Then the a -number of $X = C_1 \times C_2$ satisfies*

$$a(C_1 \times C_2) = \#\{i : 1 \leq i \leq 2, C_i \text{ is not ordinary}\}$$

Proof. If both C_1 and C_2 are ordinary, the Frobenius map acts bijectively on $H^2(X, \mathcal{O}_X) = H^1(C_1, \mathcal{O}_{C_1}) \otimes H^1(C_2, \mathcal{O}_{C_2})$ and $a(X) = 0$. Consider for $i = 1, 2$ the Hodge filtration of de Rham cohomology

$$H_{DR}^1(C_i) = F_{(i)}^0 \supset F_{(i)}^1 \supset (0).$$

The Hodge filtration

$$H_{DR}^2(X) = F^0 \supset F^1 \supset F^2 \supset (0)$$

of de Rham cohomology $H_{DR}^2(X)$ has F^2 given by

$$F^2 = F_{(1)}^1 \otimes F_{(2)}^1.$$

We consider the Frobenius map

$$H^2(X, \mathcal{O}_X) \cong H^1(C_1, \mathcal{O}_{C_1}) \otimes H^1(C_2, \mathcal{O}_{C_2}) \xrightarrow{F \otimes F} H_{DR}^2(X)$$

If both C_i are non-ordinary, then there exists an element α_i in $H^1(C_i, \mathcal{O}_{C_i})$ such that $F(\alpha_i) = 0$ on $H^1(C_i, \mathcal{O}_{C_i})$. This means that

$$(F \otimes F)(\alpha_1 \otimes \alpha_2) \in F^2 = F_{(1)}^1 \otimes F_{(2)}^1$$

and $a(X) = 2$. If exactly one is ordinary then the image of $H_{DR}^2(X, \mathcal{O}_X)$ lies in $F^1 \cap H_{DR}^1(C_1) \otimes H_{DR}^1(C_2)$ and does not lie in $F_{(1)}^1 \otimes F_{(2)}^1$. We thus see that the a -number equals the number of non-ordinary factors.

Remark 8.2. If both C_1 and C_2 are non-ordinary, then $a(X) = 2$, and hence in this case the h -number of $C_1 \times C_2$ is equal to ∞ .

Let X_m be the Fermat surface defined in \mathbf{P}^3 by the homogeneous equation

$$z_0^m + z_1^m + z_2^m + z_3^m = 0.$$

We assume $m \geq 4$ and that m is prime to the characteristic p .

We shall calculate the a -number of X_m by using the inductive structure of Fermat varieties as employed in [15] and [17]. For this we define the Fermat curve C_m by the equation

$$x_0^m + x_1^m + x_2^m = 0.$$

By [15] and [17] we have a rational map

$$\varphi : C_m \times C_m \longrightarrow X_m$$

defined by

$$((x_0, x_1, x_2), (y_0, y_1, y_2)) \mapsto (x_0 y_2, x_1 y_2, \epsilon x_2 y_0, \epsilon x_2 y_1),$$

where ϵ is a fixed $2m$ -th root of unity with $\epsilon^m = -1$.

The rational map φ is not defined at the m^2 points where both z_2 and y_2 vanish. Let Z_m be the surface obtained by blowing up $C_m \times C_m$ at these m^2 points.

An element ζ of the group μ_m of m -th roots of unity acts on $C_m \times C_m$ via

$$((x_0, x_1, x_2), (y_0, y_1, y_2)) \mapsto ((x_0, x_1, \zeta x_2), (y_0, y_1, \zeta y_2)).$$

We set $G = \mu_m$. The fixed point set of this action is equal to the locus of indeterminacy of φ and this action naturally extends after blowing up to Z_m . We have the following diagram

$$\begin{array}{ccc} Z_m & \xrightarrow{\psi} & C_m \times C_m \\ \downarrow \tilde{\varphi} & & \downarrow \varphi \\ Z_m/G & \xrightarrow{\phi} & X_m \end{array}$$

Here, the quotient surface Z_m/G is nonsingular and ϕ contracts $2m$ non-singular rational curves. For the details of this construction we refer to [15] or [17]. We derive a diagram in cohomology

$$\begin{array}{ccccc} H^1(C_m, \mathcal{O}_{C_m}) \otimes H^1(C_m, \mathcal{O}_{C_m}) & \longrightarrow & H^2(Z_m, \mathcal{O}_{Z_m}) & & \\ & & \uparrow \tilde{\varphi}^* & & \\ & & H^2(X_m, \mathcal{O}_{X_m}) & \longrightarrow & H^2(Z_m/G, \mathcal{O}_{Z_m/G}) \end{array}$$

where the horizontal arrows are isomorphisms and the vertical map identifies the cohomology $H^2(Z_m/G, \mathcal{O}_{Z_m/G})$ with the invariants $H^2(Z_m, \mathcal{O}_{Z_m})^G$. We conclude that $H^2(X_m, \mathcal{O}_{X_m})$ equals the invariant subspace $(H^1(C_m, \mathcal{O}_{C_m}) \otimes H^1(C_m, \mathcal{O}_{C_m}))^G$.

In order to calculate the action of $G = \mu_m$ on the cohomology group $H^1(C_m, \mathcal{O}_{C_m})$, we consider the open covering $U_i = \{(x_0, x_1, x_2) \in C_m \mid x_i \neq 0\}$ with $i \in \{0, 1\}$. The functions $t_1 = x_1/x_0$ and $t_2 = x_2/x_0$ define affine coordinates on the curve U_0 given by

$$1 + t_1^m + t_2^m = 0.$$

We represent elements of $H^1(C_m, \mathcal{O}_{C_m})$ as Čech cocycles with respect to the affine open covering $\{U_0, U_1\}$. They are represented by regular functions on $U_0 \cap U_1$. We set

$$\alpha_{a,b} = t_2^b/t_1^a.$$

Lemma 8.3. *If a is not positive or $a \geq b$ then $\alpha_{a,b}$ is cohomologous to zero.*

Proof. If a is non-positive, then $\alpha_{a,b}$ is regular on U_0 and the co-cycle $\omega = (-\alpha_{a,b}, 0) \in \Gamma(U_0, 0_{C_m}) \oplus \Gamma(U_1, 0_{C_m})$ gives $\delta(\omega) = \alpha_{a,b}$. Similarly, if $a \geq b$, then $\alpha_{a,b}$ is regular on U_1 ; then the co-cycle $\omega = (0, \alpha_{a,b}) \in \Gamma(U_0, 0_{C_m}) \oplus \Gamma(U_1, 0_{C_m})$ gives $\delta(\omega) = \alpha_{a,b}$. \square

We let

$$\Xi = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq a < b \leq m - 1\}.$$

Proposition 8.4. *A basis of $H^1(C_m, O_{C_m})$ is given by the set of cocycles*

$$\{\alpha_{a,b} = t_2^b/t_1^a : (a, b) \in \Xi\}.$$

Proof. Note that the cardinality of this set equals the dimension of $H^1(C_m, O_{C_m})$. So it suffices to show that these co-cycles generate this cohomology group. A regular function f on $U_0 \cap U_1$ can be considered as a rational function on U_0 with poles only at the points given by $t_1 = 0$. So f is a linear combination of the functions $\alpha_{c,d}$ with c an integer and d a non-negative integer. By Lemma 8.3 we can assume $c \geq 1$. If $d \geq m$, then we have

$$\alpha_{c,d} = t_2^{d-m}t_2^m/t_1^c = t_2^{d-m}(-1 - t_1^m)/t_1^c = -\alpha_{c,d-m} - \alpha_{c-m,d-m}$$

showing that we can assume $d \leq m - 1$. \square

The action of $\zeta \in G = \mu_m$ on $H^1(C_m, O_{C_m})$ is obviously given by

$$\alpha_{a,b} \mapsto \zeta^b \alpha_{a,b}.$$

We write our prime as

$$p = d + nm \quad \text{with } 1 \leq d \leq m - 1.$$

For a integer ℓ , if $\ell \equiv k \pmod{m}$ with $0 \leq k \leq m - 1$, we set $\bar{\ell} = k$. Using this notation, we have $d = \bar{p}$. Then, by Lemma 8.3 the Frobenius action on $H^1(C_m, O_{C_m})$ is given by

$$\begin{aligned} F^* \alpha_{a,b} &= t_2^{bp}/t_1^{ap} = t_2^{bd}(t_2^m)^{nb}/t_1^{ad}t_1^{mna} \\ &= t_2^{bd}(-t_1^m - 1)^{nb}/t_1^{ad}t_1^{mna} \\ &= (-1)^{nb}t_2^{bd}(\sum_{i=0}^{nb} \binom{nb}{i} t_1^{(i-an)m})/t_1^{ad} \end{aligned}$$

and this last expression is cohomologous to

$$(-1)^{nb}t_2^{\overline{bd}}/t_1^{\overline{ad}}.$$

We know by Lemma 8.3 that $(-1)^{nb}t_2^{\overline{bd}}/t_1^{\overline{ad}}$ is zero in $H^1(C_m, O_{C_m})$ if and only if $\overline{bd} \leq \overline{ad}$. Using this result (or by Koblitz [11]), we get the following lemma.

Lemma 8.5. *The Frobenius map induces a p -th linear isomorphism on $H^1(C_m, O_{C_m})$ if and only if $p \equiv 1 \pmod{m}$. The Frobenius map induces the zero map on $H^1(C_m, O_{C_m})$ if and only if $p \equiv -1 \pmod{m}$.*

Remark 8.6. By this lemma and Ibukiyama-Katsura-Oort [8], for $p \equiv -1 \pmod{m}$ the Jacobian variety of C_m is isomorphic to a product of supersingular elliptic curves.

Theorem 8.7. *Let m be an integer ≥ 4 and prime to p . Then the a -number of the Fermat surface X_m is given by*

$$a(X_m) = \begin{cases} 0 & p \equiv 1 \pmod{m} \\ 1 & p \equiv 2 \text{ or } 2^{-1} \text{ in } (\mathbb{Z}/m\mathbb{Z})^* \\ 2 & \text{otherwise} \end{cases}$$

Proof. Write $C = C_m$ and $X = X_m$ and let

$$H_{dR}^1(C) = F_C^0 \supset F_C^1 \supset \{0\}$$

be the Hodge filtration of $H_{dR}^1(C)$. Since the image of $H^2(X, O_X)$ lands in the G -invariant part of $H_{dR}^1(C) \otimes H_{dR}^1(C)$ in $H_{dR}^2(X)$ we consider the Hodge filtration on $(H_{dR}^1(C) \otimes H_{dR}^1(C))^G$

$$(H_{dR}^1(C) \otimes H_{dR}^1(C))^G = F^0 \supset F^1 \supset F^2 \supset \{0\}$$

Then we have

$$F^1 = \{F_C^1 \otimes F^0 + F^0 \otimes F_C^1\}^G \quad \text{and} \quad F^2 = \{F_C^1 \otimes F_C^1\}^G.$$

Thus by Lemma 8.5 we see that $a(X_m) = 0$ if and only if $p \equiv 1 \pmod{m}$. Moreover, if $p \equiv -1 \pmod{m}$ then $a(X_m) = 2$. From here on we assume $p \not\equiv \pm 1 \pmod{m}$, i.e., $2 \leq d \leq m-2$. In this case either $a(X_m) = 1$ or $a(X_m) = 2$. We look for a G -invariant element

$$\gamma = \gamma_{a,a',b} = \alpha_{a,b} \otimes \alpha_{a',m-b}$$

with (a, b) and $(a', m-b)$ in Ξ such that $F^* \alpha_{a,b} = 0$, $F^* \alpha_{a',m-b} = 0$ in $H^1(C_m, O_{C_m})$. If such a γ exists, then the Frobenius image of γ in $H_{dR}^2(X_m)$ is contained in F^2 and so we have $a(X_m) = 2$. If such an element doesn't exist, then we see $a(X_m) = 1$. Hence we are reduced to the following combinatorial problem: $a(X_m) = 2$ if and only if the set $Y(m, d)$

$$\{(a, a', b) : 1 \leq a, a', b \leq m-1; a < b, a' < m-b, \overline{da} \geq \overline{db}, \overline{da'} \geq \overline{d(m-b)}\}$$

is not empty.

We first deal with the case $d = 2$. This implies that m is odd. If $a < b$ and $\overline{2b} \leq \overline{2a}$ then we have $(m+1)/2 \leq b \leq (m-1-a)/2$. Similarly, if $a' < m-b$ and $\overline{2(m-b)} \leq \overline{2a'}$ then $(m+1)/2 \leq m-b \leq (m-1-a')/2$; but then $m = b + (m-b) \geq m+1$, a contradiction. Thus $a(X_m) = 2$ in this case.

We set $\ell = [m/d]$. We shall assume that $d > 2$. If $\ell > 3$ or $\ell = 2$ then the element $(\ell, [(d-2)m/d], \ell+2)$ is in $Y(m, d)$ or (if $\ell = 2$ and $r < d/2$) the element $(2, [(d-2)m/d], 3)$ is in $Y(m, d)$.

Suppose now that $\ell = 1$ and $m \neq 2d-1$. Since $\gcd(d, m) = 1$ we have integers x, y with $y > 0$ such that $xd + ym = 1$. Put $b = \overline{1-x}$. We then can write $bd = d-1 + ym$. Then the element $(b-1, [(d-y-1)m/d], b)$ is in $Y(m, d)$ as the reader may check. Finally, if $p \equiv 2^{-1} \pmod{m}$, i.e., $m = 2d-1$, consider the interval $[il, (i+1)\ell]$. It contains two integers, say $b, b+1$. Suppose we have an a such that $1 \leq a < b$ with $\overline{da} \geq \overline{db}$. Then $m-b$ is the larger integer in the interval $[j\ell, (j+1)\ell]$. Therefore if $a' < m-b$ we will have $\overline{da'} < \overline{d(m-b)}$ and the set $Y(m, d)$ is empty. \square

The reader may check that $Y(m, d)$ and $Y(m, d')$ for $dd' = 1 \pmod{m}$ have the same cardinality; this shortens the proof.

Corollary 8.8. *For the Fermat surface X_m in characteristic p (with $m \geq 4$, $\gcd(m, p) = 1$) the height $h(X_m)$ equals 1 if and only if $p \equiv 1 \pmod{m}$. Furthermore, If $p \not\equiv 1, 2, 2^{-1} \pmod{m}$ we have $h(X_m) = \infty$.*

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